APPROXIMATE HOMOMORPHISMS OF TERNARY SEMIGROUPS

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ABSTRACT. A mapping $f : (G_1, [\ ]_1) \to (G_2, [\ ]_2)$ between ternary semigroups will be called a ternary homomorphism if $f([xyz]_1) = [f(x)f(y)f(z)]_2$. In this paper, we prove the generalized Hyers–Ulam–Rassias stability of mappings of commutative semigroups into Banach spaces. In addition, we establish the superstability of ternary homomorphisms into Banach algebras endowed with multiplicative norms.

1. INTRODUCTION

Ternary algebraic operations were considered in the XIX-th century by several mathematicians such as A. Cayley [7] who introduced the notion of "cubic matrix" which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 ([13]). The simplest example of such non-trivial ternary operation is given by the following composition rule:

$$\{a, b, c\}_{ijk} = \sum_{l,m,n} a_{nil} b_{ljm} c_{mkn}, \quad i, j, k... = 1, 2, ..., N$$

Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their possible applications in physics. Some significant physical applications are as follows (see [14, 15]).

(i) The algebra of "*nonions*" generated by two matrices:

$$\eta_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$\eta_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{bmatrix},$$

where $\omega = e^{2\pi i/3}$, was introduced by Sylvester [21] as a ternary analog of Hamilton's quaternions; cf. [1].

(ii) The quark model inspired a particular brand of ternary algebraic systems. The so-called "*Nambu mechanics*" is based on such structures (see also [23, 5]). Quarks apparently couple by packs of 3.

(iii) A natural ternary composition of 4-vectors in the 4-dimensional Minkowskian space-time M_4 can be defined as an example of a ternary operation:

$$(X, Y, Z) \rightarrow U(X, Y, Z) \in M_4,$$

with the resulting 4-vector U^μ defined via its components in a given coordinate system as follows:

$$U^\mu(X, Y, Z) = g^{\mu\sigma}\eta_{\sigma\nu\lambda\rho}X^\nu Y^\lambda Z^\rho \qquad \mu, \nu, \cdots = 0, 1, 2, 3,$$

where $g^{\mu\sigma}$ is the metric tensor, and $\eta_{\sigma\nu\lambda\rho}$ is the canonical volume element of M_4 (see [14]).

There is also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the "anyons"), supersymmetric theories, Yang-Baxter equation, etc.; cf. [1, 14, 25].

Following the terminology of [6], a non-empty set G with a *ternary* operation $[\,] : G \times G \times G \rightarrow G$ is called *ternary groupoid* and is denoted by $(G, [\,])$. The ternary groupoid $(G, [\,])$ is called commutative if $[x_1 x_2 x_3] = [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]$ for all $x, y, z \in G$ and all permutation σ of $\{1, 2, 3\}$.

If a binary operation $\odot$ is defined on G such that $[xyz] = (x \odot y) \odot z$ for all $x, y, z \in G$, then we say that $[\,]$ is derived from $\odot$. Note that the ternary semigroup G of all odd polynomials in one variable equipped with the ternary operation $[p_1 p_2 p_3] = p_1 \diamond p_2 \diamond p_3$, where $\diamond$ denotes the usual multiplication of polynomials, is not closed under the binary operation $\diamond$.

We say that $(G, [\,])$ is a *ternary semigroup* if the operation $[\,]$ is *associative*, i.e. if $[[xyz]\,uv] = [x\,[yzu]\,v] = [xy\,[zuv]]$ holds for all $x, y, z, u, v \in G$ (see also [4]). We shall write x^3 instead of $[xxx]$.

A mapping $f : (G_1, [\ ]_1) \to (G_2, [\ ]_2)$ between ternary groupoids will be called a *ternary homomorphism* if $f([xyz]_1) = [f(x)f(y)f(z)]_2$. For instance, let us define $f : H \to H$ by $f(p) = -\mathbf{i}p$ where $\mathbf{i}$ is the imaginary unit and H denotes the set of all polynomials in one variable with coefficients in $\mathbb{C}$ equipped with the usual multiplication of polynomials. Then f is a ternary homomorphism but it is not a homomorphism in the binary mode.

Suppose that we are given a functional equation $E(f) = 0$ $(\mathcal{E})$ and we are in a framework where the notion of boundedness of f and $E(f)$ makes sense, furthermore, we assume that $E(f)$ is bounded whenever f is bounded. We say the functional equation $(\mathcal{E})$ is *superstable* if the boundedness of $E(f)$ implies that either f is bounded or $E(f) = 0$. This notion is "stronger" than the concept of *stability* in the sense that we say the functional equation $(\mathcal{E})$ is stable if any function g satisfying the equation $(\mathcal{E})$ approximately is near to a true solution of $(\mathcal{E})$. In particular one may consider approximate ternary semigroup homomorphisms.

The stability of functional equations was first investigated by S. M. Ulam [24] in 1940. More precisely, he proposed the following problem:

Given a group G_1, a metric group (G_2, d) and a positive number ϵ, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in G_1$?

As mentioned above, when this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, D. H. Hyers [11] gave a partial solution of Ulam's problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th. M. Rassias [18] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. This phenomenon of stability that was introduced by Th. M. Rassias [18] is called the Hyers–Ulam–Rassias stability. In 1992, a generalization of Rassias' theorem was obtained by Găvruta [10].

During the last decades several stability problems of functional equations have been investigated be many mathematicians. A large list of references concerning the stability of functional equations can be found in [2, 8, 12, 16, 17].

In the following section, using a sequence of Hyers type, we prove the generalized Hyers–Ulam–Rassias stability of ternary homomorphisms from commutative ternary semigroups into Banach spaces. In the third section, we follow the strategy of [3] to establish the superstability of ternary semigroup homomorphisms into Banach algebras endowed with

multiplicative norms. In this section, applying the method of [22], we provide another supersatbility result concerning complex-valued ternary semigroup homomorphisms. For an extensive account on approximately homomorphisms in the binary mode we refer the reader to [19]. This paper may be regarded as a continuation of the investigation of ternary semigroups and their applications; see [20].

2. Generalized Hyers-Ulam-Rassias Stability

The most famous method which has been widely applied to establish the stability of functional equations is the "direct method" based on an iteration process; see [2, 16].

Theorem 2.1. *Let G be a ternary semigroup, X be a Banach space and let $\varphi : G \times G \times G \to [0, \infty)$ be a function such that*

$$\widetilde{\varphi}(x, y, z) := \frac{1}{3} \sum_{n=0}^{\infty} 3^{-n} \varphi(x^{3^n}, y^{3^n}, z^{3^n}) < \infty.$$

Suppose that $f : G \to X$ is a mapping satisfying

(2.1) $$\|f([xyz]) - f(x) + f(y) + f(z)\| \le \varphi(x, y, z),$$

for all $x, y, z \in G$. Then there exists a unique mapping $T : G \to X$ such that

$$\|f(x) - T(x)\| \le \widetilde{\varphi}(x, x, x),$$

and $T(x^3) = 3T(x)$ for all $x \in G$. If G is commutative, then T is a ternary homomorphism.

Proof. Putting $y = z = x$ in inequality (2.1) we get

$$\|f(x^3) - 3f(x)\| \le \varphi(x, x, x).$$

By induction, one can show that

(2.2) $$\|3^{-n} f(x^{3^n}) - f(x)\| \le \frac{1}{3} \sum_{k=0}^{n-1} 3^{-k} \varphi(x^{3^k}, x^{3^k}, x^{3^k}),$$

for all $x \in G$ and for all positive integer n, and

$$\|3^{-n} f(x^{3^n}) - 3^{-m} f(x^{3^m})\| \le \frac{1}{3} \sum_{k=m}^{n-1} 3^{-k} \varphi(x^{3^k}, x^{3^k}, x^{3^k}),$$

for all $x \in G$ and for all nonnegative integers m, n with $m < n$. Hence $\{3^{-n} f(x^{3^n})\}$ is a Cauchy sequence in X. Due to the completeness of X we conclude that this sequence is convergent. Set now

$$T(x) = \lim_{n \to \infty} 3^{-n} f(x^{3^n}), \qquad x \in G.$$

Hence

$$T(x^3) = \lim_{n \to \infty} 3^{-n} f(x^{3^{n+1}}) = 3 \lim_{n \to \infty} 3^{-(n+1)} f(x^{3^{n+1}}) = 3T(x),$$

for all $x \in G$. If $n \to \infty$ in inequality (2.2), we obtain

$$\|f(x) - T(x)\| \leq \widetilde{\varphi}(x, x, x),$$

for all $x \in G$.

Next, assume that G is commutative. Replace x by x^{3^n}, y by y^{3^n} and z by z^{3^n} in inequality (2.1) and divide both sides by 3^n to obtain the following

$$\|3^{-n} f([xyz]^{3^n}) - 3^{-n} f(x^{3^n}) - 3^{-n} f(y^{3^n}) - 3^{-n} f(z^{3^n})\| \leq 3^{-n} \varphi(x^{3^n}, x^{3^n}, x^{3^n}).$$

Let n tend to infinity. Then

$$T([xyz]) = T(x) + T(y) + T(z),$$

for all $x, y, z \in G$.

If T' is another mapping with the required properties, then

$$
\begin{aligned}
\|T(x) - T'(x)\| &= \frac{1}{3^n} \|3^n T(x) - 3^n T'(x)\| \\
&= \frac{1}{3^n} \|T(x^{3^n}) - T'(x^{3^n})\| \\
&\leq \frac{1}{3^n} (\|T(x^{3^n}) - f(x^{3^n})\| + \|f(x^{3^n}) - T'(x^{3^n})\|) \\
&\leq \frac{2}{3^n} \widetilde{\varphi}(x, x, x).
\end{aligned}
$$

Passing to the limit as $n \to \infty$ we get $T(x) = T'(x), x \in G$. $\qquad \square$

Corollary 2.2. *Let G be a ternary semigroup, X be a Banach space and $\epsilon > 0$. Suppose that $f : G \to X$ is a mapping satisfying*

$$\|f([xyz]) - f(x)f(y)f(z)\| \leq \epsilon,$$

for all $x, y, z \in G$. Then there exists a unique mapping $T : G \to X$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2}\epsilon,$$

and $T(x^3) = 3T(x)$ for all $x \in G$. If G is commutative, then T is a ternary homomorphism.

3. SUPERSTABILITY

We start to prove the superstability of ternary semigroup homomorphisms. In the main theorem we deal with Banach algebras whose norms are multiplicative. Some examples of such algebras are provided by real (or complex) field, quaternions and Caley numbers. Our main result is as follows.

Theorem 3.1. *Suppose that $(G, [\,])$ is a ternary semigroup and A is a normed algebra whose norm is multiplicative, i.e. $\|ab\| = \|a\|\|b\|$ for all $a, b \in A$. Assume that $\epsilon \geq 0$ and $f : G \to A$ satisfy the following condition*

$$(3.1) \qquad\qquad \|f([xyz]) - f(x)f(y)f(z)\| \leq \epsilon,$$

for all $x, y, z \in G$. Then either $\|f(x)\| \leq \delta$ for all $x \in G$, where $\delta = \frac{1+\sqrt{1+4\epsilon}}{2} > 1$, or else $f([xyz]) = f(x)f(y)f(z)$ for all $x, y, z \in G$.

Proof. Choose $\delta > 1$ such that $\delta^2 - \delta = \epsilon$. Suppose that there is an element $u \in G$ such that $\|f(u)\| > \delta$. Hence $\|f(u)\| = \delta + p$ for some $p > 0$. By inequality (3.1), $\|f(u)^3 - f(u^3)\| \leq \epsilon$. Hence

$$\begin{aligned}
\|f(u^3)\| &= \|f(u)^3 - (f(u)^3 - f(u^3))\| \\
&\geq \|f(u)\|^3 - \|f(u)^3 - f(u^3)\| \\
&\geq (\delta + p)^2 - \epsilon \\
&\geq \delta + 2p.
\end{aligned}$$

Assume the induction assumption $\|f(u^{3^n})\| \geq \delta + (n+1)p$. we have

$$\begin{aligned}
\|f(u^{3^{n+1}})\| &= \|f(u^{3^n})f(u^{3^n})f(u^{3^n}) - (f(u^{3^n})f(u^{3^n})f(u^{3^n}) - f(u^{3^n}.u^{3^n}.u^{3^n}))\| \\
&\geq \|f(u^{3^n})\|^3 - \|f(u^{3^n})^3 - f((u^{3^n})^3)\| \\
&\geq (\delta + (n+1)p)^2 - \delta^2 + \delta \\
&\geq \delta + (n+2)p.
\end{aligned}$$

Therefore $\|f(u^{3^n})\| \geq \delta + (n+1)p$ holds for all positive integer n.

Now, let $x, y, z, t, s \in G$. We have

$$\|f([[xyz]ts]) - f([xyz])f(t)f(s)\| \leq \epsilon,$$

and

$$\|f([x[yzt]s]) - f(x)f([yzt])f(s)\| \leq \epsilon,$$

so that

$$\|f([xyz])f(t)f(s) - f(x)f([yzt])f(s)\| \leq 2\epsilon,$$

hence

$$
\begin{aligned}
\|f([xyz]) - f(x)f(y)f(z)\|\|f(t)\|\|f(s)\| &= \|f([xyz])f(t)f(s) - f(x)f(y)f(z)f(t)f(s)\| \\
&\leq \|f([xyz])f(t)f(s) - f(x)f([yzt])f(s)\| \\
&\quad + \|f(x)f([yzt])f(s) - f(x)f(y)f(z)f(t)f(s)\| \\
&\leq 2\epsilon + \|f(x)\|\epsilon\|f(s)\|.
\end{aligned}
$$

Replacing both s and t by u^{3^n}, we obtain

$$
\begin{aligned}
\|f([xyz]) - f(x)f(y)f(z)\| &\leq \frac{2\epsilon}{\|f(u^{3^n})\|^2} + \frac{\epsilon\|f(x)\|}{\|f(u^{3^n})\|} \\
&\leq \frac{2\epsilon}{(\delta + (n+1)p)^2} + \frac{\epsilon\|f(x)\|}{(\delta + (n+1)p)}.
\end{aligned}
$$

Letting n tend to infinity, we obtain $f([xyz]) = f(x)f(y)f(z)$. $\qquad\square$

As the following example shows, the theorem fails if the algebra does not have the multiplicative norm property. This example is due to J. Baker [3].

Example 3.2. Given $\epsilon > 0$, choose δ such that $|\delta - \delta^3| = \epsilon$. Define f from $\mathbb{R}$ into the algebra $M_2(\mathbb{R})$ of 2×2 matrices with real elements and the operator norm by

$$f(x) = \begin{bmatrix} e^x & 0 \\ 0 & \delta \end{bmatrix}$$

. Then $\|f(x + y + z) - f(x)f(y)f(z)\| = \epsilon$ for all $x, y, z \in \mathbb{R}$ while f is unbounded and does not fulfill $f(x + y + z) = f(x)f(y)f(z)$. Here we assume $\mathbb{R}$ as the ternary semigroup derived from the binary $+$.

Let $(G, [.])$ be a ternary semigroup, K be a field. For $y, z \in G$ and $\varphi : G \to K$ we define the right translation of φ along y, z by $\varphi_{y,z}(x) = \varphi([xyz])$. Suppose that V is a vector space of the K-valued functions on G. We say V is right invariant if whenever $\varphi \in V$, the function $\varphi_{y,z}$ belongs to V for all $y, z \in G$.

Lemma 3.3. *Let $(G, [\ ])$ be a ternary semigroup, K be a field, V be a right invariant vector space of K-valued functions on G, and let $\varphi, f : G \to K$ be nonzero functions such that the function $x \mapsto \varphi([xyz]) - \varphi(x)f(y)f(z)$ belongs to V for each $y, z \in G$. Then either $\varphi \in V$ or f is a ternary homomorphism.*

Proof. Assume that f is not a ternary homomorphism of G into K regarded as a derived ternary semigroup. Hence there are elements $y, z, w \in G$ such that $f([yzw]) - f(y)f(z)f(w) \neq 0$. Assume that $u \in G$ such that $f(u) \neq 0$. Then

$$
\begin{aligned}
\varphi([[xyz]wu]) - \varphi([xyz])f(w)f(u) &= \big(\varphi([x[yzw]u]) - \varphi(x)f([yzw])f(u)\big) \\
&\quad - \big(\varphi([xyz]) - \varphi(x)f(y)f(z)\big)f(w)f(u) \\
&\quad + \varphi(x)\big(f([yzw]) - f(y)f(z)f(w)\big)f(u),
\end{aligned}
$$

Put $\psi(x) = \varphi([xwu]) - \varphi(x)f(w)f(u)$. By the hypothesis, $\psi \in V$. Therefore

$$
\begin{aligned}
\varphi(x) &= \big(f([yzw]) - f(y)f(z)f(w)\big)^{-1}f(u)^{-1} \\
&\quad \times \psi_{y,z}(x) - \\
&\quad (\varphi([x[yzw]u]) - \varphi(x)f([yzw])f(u)) + \\
&\quad (\varphi([xyz]) - \varphi(x)f(y)f(z))f(w)f(u)).
\end{aligned}
$$

Since V is right invariant we conclude that the function appeared in the right hand side of the last equality must belong to V. $\qquad\square$

Now we are ready to give another important result as follows.

Theorem 3.4. *Let $(G, [\ ])$ be a ternary semigroup, and let $\varphi, f : G \to \mathbb{C}$ be nonzero functions for which there exists a function $\alpha : G \times G \to [0, \infty)$ such that*

$$
|\varphi([xyz]) - \varphi(x)f(y)f(z)| \leq \alpha(y, z),
$$

for all $x, y, z \in G$. Then either φ is bounded (i.e. there is $M > 0$ such that $|\varphi(x)| \leq M$ for all $x \in G$) or f is a ternary homomorphism.

Proof. Let V be the space of all bounded complex-valued functions on G and apply Lemma 3.3. $\qquad\square$

Corollary 3.5. *Let $(G, [\ ])$ be a ternary semigroup, $\epsilon > 0$, and $f : G \to \mathbb{C}$ be a nonzero function such that*

$$|f([xyz]) - f(x)f(y)f(z)| \leq \epsilon,$$

for all $x, y, z \in G$. Then either f is bounded or f is a ternary homomorphism.

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